Name:___________________________

SM3 12.3: Pythagorean Identity

A right triangle is constructed with a hypotenuse of length r and an acute angle θ.

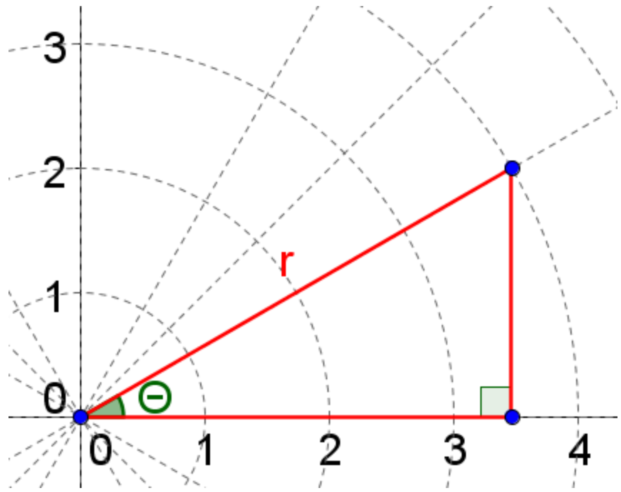
In terms of r and θ, how long are the legs of the triangle?

Use the trigonometric definitions to solve for opp and adj in terms of r and θ:

$$\sin(\theta) = \frac{opp}{hyp} \qquad \cos(\theta) = \frac{adj}{hyp}$$

$$\sin(\theta) = \frac{opp}{\quad} \qquad \cos(\theta) = \frac{adj}{\quad}$$

$$= opp \qquad = adj$$



Add the values of the lengths of the legs in terms of r and θ to the picture.

The Pythagorean Identity is derived from the Pythagorean Theorem, $a^2 + b^2 = c^2$, by substituting the lengths of the triangle in the picture into the Pythagorean Theorem.

Follow the steps of the proof to derive the Pythagorean Identity (the last statement of the proof, directly above QED, is the Pythagorean Identity.

Statements	**Reasons**
$a^2 + b^2 = c^2$	Given
$(\qquad)^2 + (\qquad)^2 = r^2$	Substitution
$\quad + \quad = r^2$	Multiplication
$\quad + \quad = 1$	Divide by r^2
QED	

Notice that because the Pythagorean Identity does not contain r, it works for any r.

Let's make the Pythagorean Identity more malleable by manipulating its terms. Complete each proof by using the Pythagorean Identity as the given and following the reasons.

$\quad + \quad = 1$	Given
	Subtract $\cos^2\theta$
QED	

$\quad + \quad = 1$	Given
	Subtract $\sin^2\theta$
QED	

We now have 3 equations that are each essentially the Pythagorean Identity. However, all of them only include $\sin\theta$ and $\cos\theta$. Let's develop some alternate versions of that include other trig functions.

Complete each exploration by using the Pythagorean Identity as the given and following the reasons.

$\sin^2\theta + \cos^2\theta = 1$	Given
$\frac{\sin^2\theta}{\quad} + \frac{\cos^2\theta}{\quad} = \frac{1}{\quad}$	Divide by $\cos^2\theta$
	Definition of tan,sec
QED	

$\sin^2\theta + \cos^2\theta = 1$	Given
$\frac{\sin^2\theta}{\quad} + \frac{\cos^2\theta}{\quad} = \frac{1}{\quad}$	Divide by $\sin^2\theta$
	Definition of cot,csc
QED	

The above identities can also move one term to the other side, giving us even more identities!

Write the conclusions of the above explorations in two other forms each by subtracting a term from the right to the left side of the equation.

$\tan^2\theta + 1 = \sec^2\theta$	Given
	Subtract 1
QED	

$1 + \cot^2\theta = \csc^2\theta$	Given
	Subtract 1
QED	

$\tan^2\theta + 1 = \sec^2\theta$	Given
	Subtract $\tan^2\theta$
QED	

$1 + \cot^2\theta = \csc^2\theta$	Given
	Subtract $\cot^2\theta$
QED	

We're up to 9 equations that are all essentially the Pythagorean Identity!

I know, as honorable math students, you're eager to start using square roots, conjugates, and other ideas to come up with dozens more. We could make hundreds of identities! We could fill the world's libraries with all sorts of unique, innovations by manipulating the Pythagorean Identity.

Sadly, we're not going to spend the rest of your lives producing more Pythagorean Identities. We're just going to memorize these 9 equations and make use of them. You may exhale a sigh of (faux) regret.

Memorize:

$\sin^2\theta + \cos^2\theta = 1$	The Pythagorean Identity
$\sin^2\theta = 1 - \cos^2\theta$	The Pythagorean Identity, solved for $\sin^2\theta$
$\cos^2\theta = 1 - \sin^2\theta$	The Pythagorean Identity, solved for $\cos^2\theta$
$\tan^2\theta + 1 = \sec^2\theta$	The Pythagorean Identity, divided by $\cos^2\theta$
$\tan^2\theta = \sec^2\theta - 1$	The Pythagorean Identity, divided by $\cos^2\theta$, then -1.
$1 = \sec^2\theta - \tan^2\theta$	The Pythagorean Identity, divided by $\cos^2\theta$, then $-\tan^2\theta$.
$1 + \cot^2\theta = \csc^2\theta$	The Pythagorean Identity, divided by $\sin^2\theta$
$\cot^2\theta = \csc^2\theta - 1$	The Pythagorean Identity, divided by $\sin^2\theta$, then -1.
$1 = \csc^2\theta - \cot^2\theta$	The Pythagorean Identity, divided by $\sin^2\theta$, then $-\cot^2\theta$.

You may abbreviate any of the above substitutions in a proof as Pyth ID.

You may use the Pyth IDs in either direction (e.g., you may replace $\sin^2\theta + \cos^2\theta$ with the number 1 or you may change the number 1 into $\sin^2\theta + \cos^2\theta$).

Example: Prove $\sin^3\theta\cos^4\theta = \cos^4\theta\sin\theta - \cos^6\theta\sin\theta$

Strategy: The right side has two terms, so we just need replace a portion of the left side by exchanging a term for two terms. We want less $\sin\theta$ power on the right side, so let's replace $\sin^2\theta$ on the left using a Pyth ID.

I'm only going to write the left side after the first line to save space.

Statements		**Reasons**
$\sin^3\theta\cos^4\theta =$	$\cos^4\theta\sin\theta - \cos^6\theta\sin\theta$	Given
$\sin^2\theta\cos^4\theta\sin\theta =$		Factor
$(1-\cos^2\theta)\cos^4\theta\sin\theta =$		Pyth ID
$\cos^4\theta\sin\theta - \cos^6\theta\sin\theta =$		Distribution

QED

One of the most important aspects of this skill will become clear next year during your study of integral calculus. Trigonometric identities allow you to alter expressions to be more suited to easier manipulation with calculus techniques.

Use two columns to prove each identity. You may use the equation as the first statement.

1)

Statements	Reasons
$2\cos^2\theta - 1 = 1 - 2\sin^2\theta$	Given

2)

Statements	Reasons
$\tan^2\theta = \dfrac{1-\cos^2\theta}{\cos^2\theta}$	Given

3) $4\sin^2\theta + 4\cos^2\theta = 4$

4) $\cos\theta - \cos^3\theta = \cos\theta\sin^2\theta$

5) $\dfrac{\cos^2\theta - 1}{\cos\theta} = -\tan\theta\sin\theta$

6) $\dfrac{\sec\theta + 1}{\tan\theta} = \dfrac{\sin\theta}{1-\cos\theta}$

7) $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$

8) $\tan^4\theta + \tan^2\theta = \sec^4\theta - \sec^2\theta$

9) $(1-\tan\theta)^2 = \sec^2\theta - 2\tan\theta$

10) $(\cos\theta - \sin\theta)^2 = 1 - 2\sin\theta\cos\theta$

11) $$\frac{\cos^2\theta}{1-\sin\ \theta}=1+\sin\theta$$

12) $$(\sec^2\theta+\csc^2\theta)-(\tan^2\theta+\cot^2\theta)=2$$

13) $$\frac{1}{1-\cos\ \theta}+\frac{1}{1+\cos\ \theta}=\frac{2}{\sin^2\theta}$$

14) $$\frac{\sec^2\theta\csc\ \theta}{\sec^2\theta+\csc^2\theta}=\sin\theta$$

15) While working late on a math homework assignment, Rachel hears a commotion coming from her closet. She opens it to find herself, literally. A future version of Rachel has come back in time!

Future Rachel says, “Rachel, I need your help. I mean I need my help. I mean our help … ok this is weird.”

Current Rachel replies, “How did you get here?” Future Rachel exclaims, “There isn’t time for that, we only have about 4 or 5 sentences of dialogue left before students lose interest in this problem!”

Future Rachel adds, “It turns out that Mr. Wytiaz was right: we have to use trigonometric identities in the calculus class in chapter 7. I have to integrate $\int \tan^4 \theta \, d\theta$ and I can’t remember how to do it. Don’t worry about the integration symbol, $\int$, or the $d\theta$, you’ll learn what those mean next year. You’ve got to help me figure out how to turn that $\tan^4 \theta$ into $\tan^2 \theta \sec^2 \theta - \sec^2 \theta + 1$. I can take it from there.”

“Go figure it out yourself… myself… ourselves,” stammers Current Rachel, “I’ve got boys to think about.”

“No, you don’t get it, Rachel,” urges Future Rachel, “Mr. Wytiaz wasn’t right because he’s a great teacher, although in retrospect, I suppose that is true. He was right because he used math to take over the planet and anyone that can’t perform integral calculus has been condemned to be recycled in the organ donation system so that useful mathematicians can live longer!”

Write a two-column proof that allows Future Rachel to keep her organs.